Workshop 6: Transient heat conduction

Attempt these problems before opening the worked solutions.

1 Heating a plate

EXERCISE
Exercise 1. Plate surface temperature

In a production facility, 4-cm-thick large plates (k = 107 $\mathrm{W/m \cdot ^\circ C}$, ρ = 8520 $\mathrm{kg/m^3}$, C_p = 381.2 $\mathrm{J/kg \cdot ^\circ C}$) that are initially at a uniform temperature of 20 $^\circ$C are heated by passing them through an oven maintained at 500 $^\circ$C. The plates remain in the oven for a period of 7 min. Taking the convection heat transfer coefficient to be h = 120 $\mathrm{W/m^2 \cdot ^\circ C}$, determine the surface temperature of the plates when they come out of the oven. Assume L = W = 1 m.

See solution for Exercise 1

2 Ball bearings before quenching

EXERCISE
Exercise 2. Maximum time in air

During the quenching process of stainless steel ball bearings, it is known that the temperature of the bearings should not reach below 850 $^\circ$C prior to entering the water to minimise the developments of surface defects. Ball bearings with diameter of 1.2 cm (ρ = 8085 $\mathrm{kg/m^3}$, C_p = 0.480 $\mathrm{kJ/kg \cdot ^\circ C}$, k = 15.1 $\mathrm{W/m \cdot ^\circ C}$ and α = 3.91 × 10^{-6} $\mathrm{m^2/s}$) are taken out of the oven at a uniform temperature of 900 $^\circ$C, and are exposed to 30 $^\circ$C air for a while before they are dropped into the water. Assuming that the heat transfer coefficient of the air is 125 $\mathrm{W/m^2}$.K, how long can the spheres be left in air before quenching?

See solution for Exercise 2

3 Inferring convection from a heated sphere

EXERCISE
Exercise 3. Centre and surface temperatures

In the Betty Crocker Cookbook, it is stated that it takes 2 h 45 minutes to roast a 3.2 kg ribs that is initially at 4.5 $^\circ$C in an oven that is maintained at 163 $^\circ$C. It is recommended that a meat thermometer be used to monitor the cooking, and the rib is considered medium-

rare when the thermometer, inserted into the centre of the thickest part of the meat, registers $60\,^{\circ}\mathrm{C}$. If the ribs is approximated as a homogeneous spherical object with ρ = $1200\,\mathrm{kg/m^3}$, C_p = 4.1 $\mathrm{kJ/kg\cdot\,^{\circ}C}$, k = 0.45 $\mathrm{W/m\cdot\,^{\circ}C}$ and $\alpha = 0.91 \times 10^{-7}\ \mathrm{m^2/s}$, determine:

a. The heat transfer coefficient at the surface of the ribs

b. The temperature at the outer surface of the ribs when it is done (medium rare)

See solution for Exercise 3

Solutions

Exercise solutions

Exercise 1. Plate surface temperature

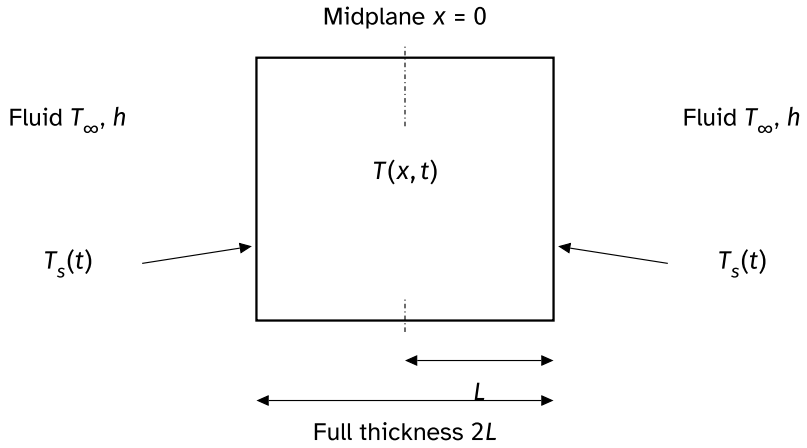


Figure 1: Both faces of the plate are heated by the oven air. Use half the plate thickness for the plane-wall solution, not its one-metre plan dimension.

Data

Quantity	Symbol	Value
Plate dimensions	$H, W, 2L$	$1.0, 1.0, 0.040\ \mathrm{m}$
Initial and oven temperatures	T_i, T_∞	$20, 500\ ^\circ\mathrm{C}$
Heating duration	t	$420\ \mathrm{s}$
Convection coefficient	h	$120\ \mathrm{W\,m^{-2}\,K^{-1}}$
Conductivity	k	$107\ \mathrm{W\,m^{-1}\,K^{-1}}$
Density	ρ	$8520\ \mathrm{kg\,m^{-3}}$
Specific heat	c_p	$381.2\ \mathrm{J\,kg^{-1}\,K^{-1}}$

Here $L = 0.020\ \mathrm{m}$ denotes half the plate thickness in the plane-wall model. The one-metre dimensions in the question set the face area.

Assumptions

- Both broad faces see the same oven temperature and coefficient.
- Edge heat transfer and radiation are neglected in the plane-wall comparison.
- Properties remain constant and the initial temperature is uniform.

Calculations

The worked solution compares lumped capacitance with the one-term plane-wall solution.

For the comparison, solve $\lambda_1 \tan \lambda_1 = Bi$ with $Bi = hL/k$ and use

$$\Theta(x,t) = A_1 e^{-\lambda_1^2 Fo} \cos\!\left(\frac{\lambda_1 x}{L}\right),$$

$$A_1 = \frac{4 \sin \lambda_1}{2\lambda_1 + \sin(2\lambda_1)},$$

$$Fo = \frac{\alpha t}{L^2}, \qquad \alpha = \frac{k}{\rho c_p}.$$

Here $x = 0$ is the centre and

$$\Theta = \frac{T - T_\infty}{T_i - T_\infty}.$$

The plate characteristic length is its volume divided by its heat-transfer area.

With only the two broad faces transferring heat,

$$V = (1)(1)(0.040) = 0.040\ \mathrm{m}^3$$

and $A_s = 2(1)(1) = 2.0\ \mathrm{m}^2$. The half-thickness and volume-to-area length are equal for this model.

Thus $L_c = V/A_s = 0.02\ \mathrm{m}$ and

$$Bi_c = \frac{hL_c}{k} = 0.02243 < 0.1.$$

Lumped capacitance is a reasonable approximation.

The uniform-temperature energy balance gives

$$\Theta = e^{-\frac{hA_s t}{\rho c_p V}}.$$

Using $t = 420\ \mathrm{s}$ gives $\Theta = 0.46029$ and

$$T = 500 - 480\Theta = 279.06\ ^\circ\mathrm{C}.$$

Within this approximation the surface and centre have the same temperature.

For the plane-wall comparison, $\alpha = 3.2945 \times 10^{-5}\ \mathrm{m^2\,s^{-1}}$ and $Fo = 34.592$.

The eigenvalue is $\lambda_1 = 0.149209$ and $A_1 = 1.003709$.

Evaluating the supplied relation at $x = L$ gives

$$\Theta_s = 1.003709\, e^{-(0.149209)^2(34.592)} \times \cos(0.149209) = 0.459504,$$

$$T_s = 279.44\ ^\circ\mathrm{C}.$$

The finite-wall result is about 0.38 degrees above the lumped estimate.

The large Fourier number makes higher eigenmodes negligible.

The temperature is not exactly uniform, but the small Biot number explains the close agreement.

Answer

The one-term plane-wall estimate gives a surface temperature of $279.4\,^{\circ}\mathrm{C}$. The simpler lumped estimate is $279.1\,^{\circ}\mathrm{C}$, consistent with the small Biot number.

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Exercise 2. Maximum time in air

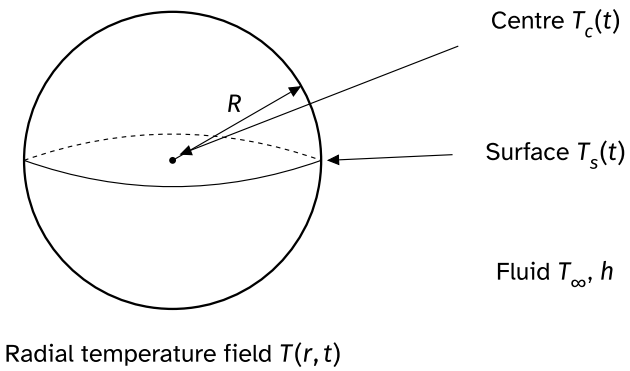


Figure 2: A small spherical bearing cools in air. Its radius is half the supplied diameter, and the exposed area is the full spherical surface.

For a sphere the volume-to-area length is one third of its radius.

With $R = 0.006\,\mathrm{m}$, $L_c = R/3 = 0.002\,\mathrm{m}$ and

$$Bi_c = \frac{hL_c}{k} = 0.01656 < 0.1.$$

Use the lumped cooling relation to solve for time:

$$t = \frac{\rho c_p R}{3h} \ln\left(\frac{T_i - T_\infty}{T - T_\infty}\right).$$

At the limiting temperature,

$$t_{\max} = \frac{8085(480)(0.006)}{3(125)} \ln\left(\frac{870}{820}\right) = 3.675\,\mathrm{s}.$$

The lumped model permits at most about $3.68\,\mathrm{s}$ in air.

This estimate treats the whole sphere as isothermal.

It does not guarantee a strict local surface limit: a real cooling sphere has a colder surface than centre.

Such a guarantee needs a spatial transient model and allowances for radiation and uncertain properties at these high temperatures.

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Exercise 3. Centre and surface temperatures

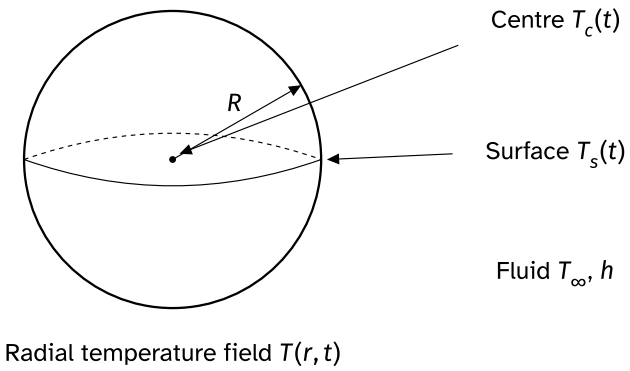


Figure 3: A spherical body has distinct centre and surface temperatures during heating. The measured centre temperature is used to infer the coefficient at the outer boundary.

Data

Quantity	Symbol	Value
Mass	m	$3.2\ \mathrm{kg}$
Density	ρ	$1200\ \mathrm{kg\,m^{-3}}$
Initial and oven temperatures	T_i, T_∞	$4.5, 163\ ^\circ\mathrm{C}$
Observed centre temperature	T_c	$60\ ^\circ\mathrm{C}$
Elapsed time	t	$9900\ \mathrm{s}$
Conductivity	k	$0.45\ \mathrm{W\,m^{-1}\,K^{-1}}$
Supplied diffusivity	α	$0.91 \times 10^{-7}\ \mathrm{m^2\,s^{-1}}$

Assumptions

Use the given rounded diffusivity for the transient calculation. Assume radial conduction, constant properties, no evaporation, no generation, and no separate radiative heat flux. The effective coefficient describes the stated idealised boundary, not a measured material property.

Calculations

The first step is to infer the radius from

$$\frac{m}{\rho} = \frac{4\pi R^3}{3}.$$

For $Bi_R = hR/k$, the positive eigenvalues satisfy $1 - \lambda_n \cot \lambda_n = Bi_R$. The full spherical solution is

$$\begin{aligned} \Theta(r,t) &= \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2 Fo} f_n(r), \\ A_n &= \frac{4(\sin\lambda_n - \lambda_n \cos\lambda_n)}{2\lambda_n - \sin(2\lambda_n)}, \\ f_n(r) &= \frac{\sin\left(\frac{\lambda_n r}{R}\right)}{\frac{\lambda_n r}{R}}, \\ Fo &= \frac{\alpha t}{R^2}. \end{aligned} \tag{1}$$

Use $f_n(0) = 1$. Retain enough terms for convergence; the usual one-term criterion $Fo > 0.2$ may not hold here. The first eigenvalue lies between zero and π, with each subsequent one between adjacent multiples of π.

The mass and density fix the radius without a geometric guess.

$$V = \frac{3.2}{1200} = 0.0026667\,\mathrm{m}^3$$

and $R = 0.0860254\,\mathrm{m}$.

With $t = 9900\,\mathrm{s}$, $Fo = 0.121737$.

The observed centre temperature requires

$$\Theta_0 = \frac{60 - 163}{4.5 - 163} = 0.649842.$$

For a trial Bi_R, solve the eigenvalue equation in each interval and evaluate Eq. 1 at the centre.

Adjust Bi_R until the sum equals 0.649842.

The following trials show how the inverse calculation brackets the measured centre temperature. Each row uses its own eigenvalues in the same series.

Trial Bi_R	First eigenvalue λ_1	Centre ratio Θ_0	Predicted centre temperature
10.0	2.83630	0.690587	53.54 °C
20.0	2.98572	0.643649	60.98 °C
17.8639	2.96741	0.649842	60.00 °C

The first trial heats the centre too slowly; the second heats it too quickly. Solving between them gives the fitted Biot number. Once it is fitted, use that same boundary coefficient to predict the surface temperature.

Five terms already agree with ten terms to better than 10^{-10} in the temperature ratio at the fitted coefficient.

This gives

$$Bi_R = 17.8639 \qquad h = \frac{Bi_R k}{R} = 93.45\,\mathrm{W\,m^{-2}\,K^{-1}}$$

The fitted spherical series gives a surface ratio of 0.0408906.

At $r = R$, retain the same eigenvalues and coefficients but multiply each term by $\sin\lambda_n/\lambda_n$.

Therefore

$$T_s = 163 + (4.5 - 163)(0.0408906) = 156.52\,^{\circ}\mathrm{C}.$$

The surface is much hotter than the centre during heating.

Lumped capacitance is unsuitable because $Bi_c = Bi_R/3 = 5.95$.

A one-term inverse calculation also introduces avoidable error at this Fourier number.

The handwritten source's trial value $Bi_R \approx 30$ does not reproduce the stated centre temperature with a converged series; its coefficient is not used as the answer.

Answer

The fitted effective coefficient is $h = 93.5\,\mathrm{W\,m^{-2}\,K^{-1}}$ and the surface temperature is $156.5\,^{\circ}\mathrm{C}$. These values use the converged spherical series and the stipulated constant-property model.

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