Heat transfer modes and steady conduction

1 Reading list

- Çengel and Ghajar (2015), sections 3.1 to 3.4.
- Bird et al. (2006), sections 10.1, 10.2 and 11.1.

2 Choose a steady-conduction model

Given conditions	Model and checks
Plane layers, fixed end temperatures or surface films	Series resistance, Eq. 1; include each exposed film.
Heat can cross separate side-by-side paths	Combine each series path before putting paths in parallel, Eq. 4.
Hollow cylinder or sphere	Use the radial resistance, Eq. 5 or Eq. 7; area varies with radius.
Internal heat generation	Use the source term in the energy balance and solve for the temperature profile, Eq. 11.

3 Heat transfer mechanisms

3.1 Example: a cup of hot tea

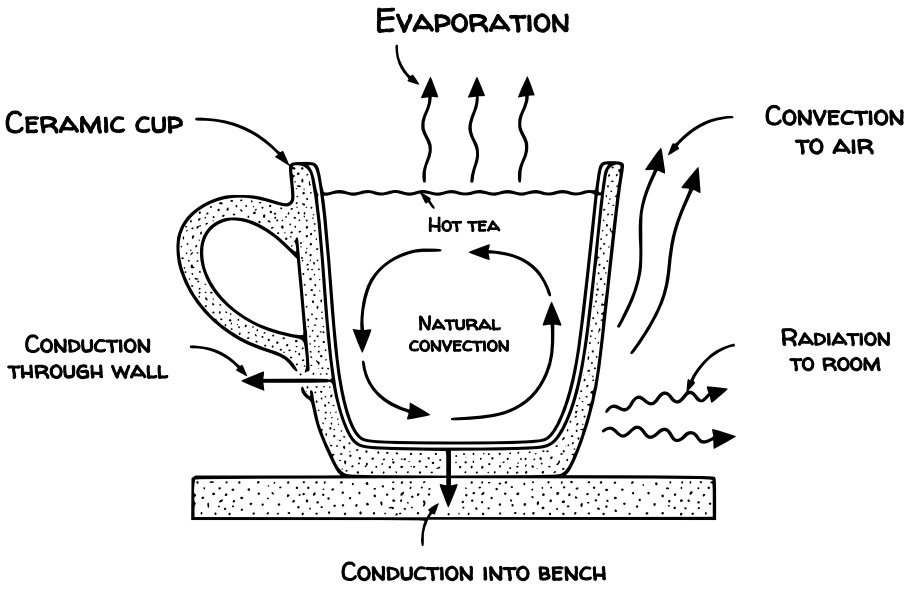


Figure 1: Section through a ceramic cup of hot tea. Arrows show circulation in the tea, rising warm air outside, conduction through the wall and into the bench, radiation to the room, and evaporation from the free surface.

A cup of hot tea cools because it loses heat to its surroundings. Let's assume the air, bench and room surfaces are cooler than the tea.

There are several modes of heat transfer at work here. Heat passes through the ceramic wall by conduction. The tea near the wall cools, becomes denser and sinks. Warmer tea rises through the middle to replace it, forming the circulation shown in Figure 1. Outside the cup, air in contact with the wall warms and rises. In both cases, temperature differences cause density differences that drive the fluid motion. This phenomenon is called natural convection.

The cup also loses heat by radiation to the cooler room surfaces. So, at the outer wall, convection and radiation occur at the same time. There are other ways for heat to leave the tea as well. Heat passes through the base into the bench by conduction. At the exposed tea surface, water evaporates into the air and takes energy with it. Evaporation therefore involves both heat and mass transfer.

Let's follow the heat leaving through the side wall. It first passes from the tea to the inner wall by convection, then through the ceramic by conduction. From the outer wall, it leaves by convection to the air and radiation to the room surfaces. These steps help explain why we use thermal resistances in series and in parallel. Heat must pass through the inner convection resistance and the wall resistance in sequence, so these are in series. At the outer wall, convection and radiation provide parallel routes for heat loss. Heat leaving through the tea surface or the base follows separate routes.

How do we calculate the heat transferred? For one-dimensional conduction, we use Fourier's law, $q = -k\,dT/dx$. Here q is the heat transferred per unit area per unit time, or heat flux,

measured in $\mathrm{W\,m^{-2}}$. The thermal conductivity k has units of $\mathrm{W\,m^{-1}\,K^{-1}}$. The minus sign tells us that heat flows towards lower temperature.

For convection, we use $q = h(T_s - T_\infty)$. Here T_s is the surface temperature and T_∞ is the fluid temperature away from the surface. The heat transfer coefficient h has units of $\mathrm{W\,m^{-2}\,K^{-1}}$. It depends on the fluid, the geometry and how the fluid flows. For example, blowing air over the cup changes h and can increase the rate of heat loss.

DEFINITION
Conduction

Conduction is the transfer of thermal energy within a medium due to a temperature gradient, through microscopic interactions without bulk motion of the medium. It occurs in solids, liquids and gases.

DEFINITION
Convection

Convection is the transfer of heat between a surface and a fluid in motion when they are at different temperatures. It involves the combined effects of conduction and bulk fluid motion.

DEFINITION
Radiation

Thermal radiation is the emission of energy as electromagnetic waves by matter at a nonzero absolute temperature. Energy transfer by radiation does not require a material medium.

DEFINITION
Heat flux

Heat flux is the rate of heat transfer per unit area normal to the direction of transfer. Its SI unit is $\mathrm{W\,m^{-2}}$. A signed heat flux specifies the direction of transfer relative to the chosen coordinate or surface normal.

DEFINITION
Thermal conductivity

Thermal conductivity, k, is a material property that relates conductive heat flux to the temperature gradient through Fourier's law. For one-dimensional conduction, $q = -k\,dT/dx$. Its SI unit is $\mathrm{W\,m^{-1}\,K^{-1}}$.

DEFINITION
Heat transfer coefficient

The heat transfer coefficient, h, is the proportionality coefficient between the convective heat flux at a surface and the temperature difference between the surface and the fluid away from it, $q = h(T_s - T_\infty)$. Its SI unit is $\mathrm{W\,m^{-2}\,K^{-1}}$. It depends on the fluid properties, surface geometry and flow conditions.

QUIZ
1. Heat transfer modes
Consider these three cases:

1. Heat passes across a layer of stationary air, with radiation neglected.
2. Water flows over a hot plate and removes heat from it.
3. Heat passes between two surfaces separated by a vacuum.

Which sequence gives the heat transfer mode in each case, in the order listed?

1. Convection, conduction, radiation.
2. Conduction, convection, radiation.
3. Conduction, radiation, convection.
4. Radiation, convection, conduction.

See solution for Quiz 1

QUIZ
2. Conductivity and the heat transfer coefficient
A fan blows air over a metal plate. Suppose we turn up the fan so the air flows faster, while keeping the plate and air temperatures the same. What happens to the plate's thermal conductivity k and the heat transfer coefficient h?

1. Both k and h stay the same because the temperatures have not changed.
2. k increases, while h stays the same.
3. k stays the same, while h generally increases.
4. Both k and h increase as the air flows faster.

See solution for Quiz 2

4 Case 1: 1D steady conduction without heat generation

4.1 A wall with several layers

DEFINITION
Thermal resistance

Thermal resistance, R, is the ratio of the temperature difference across a heat-transfer process to the resulting heat-transfer rate, $R = \Delta T / \dot{Q}$. Its SI unit is $\mathrm{K\,W^{-1}}$.

Let's consider the three-layer wall in Figure 2, with hot fluid on one side and cold fluid on the other. The fluid temperatures are T_h and T_c, with $T_h > T_c$. Each layer has the same area A, but its thickness L_i and thermal conductivity k_i may differ.

Assume steady, one-dimensional conduction through the wall, with no heat generation. The conductivity of each layer is constant, and the layers are in perfect thermal contact.

Heat first passes from the hot fluid to the wall by convection. It then conducts through each layer before leaving the wall by convection to the cold fluid. At steady state, heat enters and leaves each layer at the same rate. If heat entered a layer faster than it left, the layer would store energy and warm up.

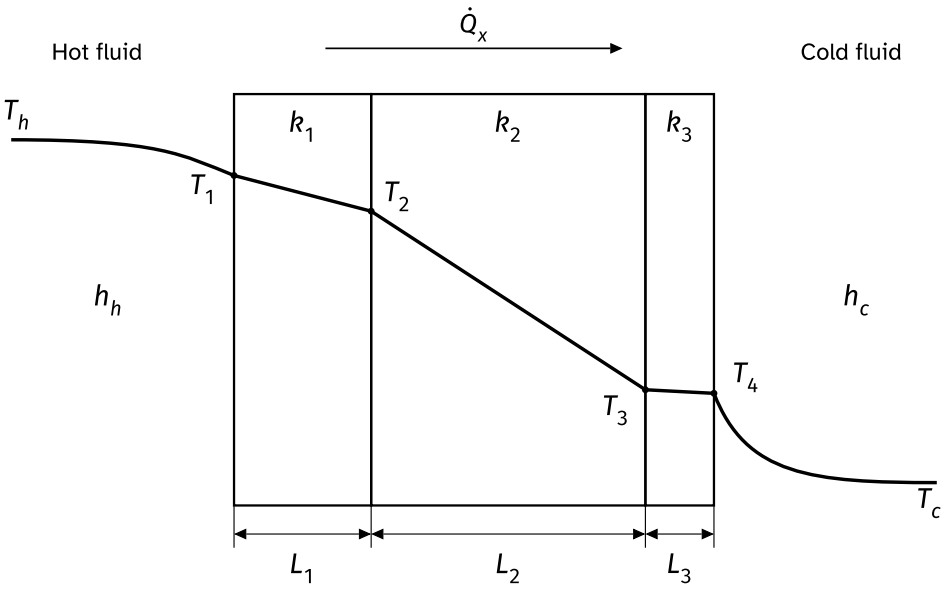


Figure 2: Schematic temperature profile from hot fluid to cold fluid through three solid layers, with surface and interface temperatures, conductivities and thicknesses labelled.

The wall surfaces are at T_1 and T_4, and the two interfaces are at T_2 and T_3. The heat transfer coefficients on the hot and cold sides are h_h and h_c, respectively. We can write the heat rate through each part as

Heat-transfer step	Heat rate
Hot fluid to wall	$\dot{Q}_x = h_h A(T_h - T_1)$
Through solid layer i	$\dot{Q}_x = \dfrac{k_i A}{L_i}(T_i - T_{i+1})$
Wall to cold fluid	$\dot{Q}_x = h_c A(T_4 - T_c)$

The middle equation applies to all three solid layers, with $i = 1, 2, 3$. Heat passes through each layer and fluid film at the same rate $\dot{Q}_x$. The temperature drop across each layer depends on its thermal resistance.

Rearranging the three heat-rate equations above to express their temperature differences gives

$$T_h - T_1 = \frac{\dot{Q}_x}{h_h A},$$

$$T_i - T_{i+1} = \frac{\dot{Q}_x L_i}{k_i A}$$

and

$$T_4 - T_c = \frac{\dot{Q}_x}{h_c A}.$$

Each has the form $\Delta T = \dot{Q}_x R$, so the factor multiplying $\dot{Q}_x$ is the thermal resistance:

$$R_h = \frac{1}{h_h A}, \qquad R_i = \frac{L_i}{k_i A}, \qquad R_c = \frac{1}{h_c A} \tag{1}$$

The convection resistances R_h and R_c are often called film resistances. They represent heat transfer between each fluid and the adjacent wall surface.

Now add the temperature drops from the hot fluid to the cold fluid:

$$(T_h - T_1) + (T_1 - T_2) + (T_2 - T_3) + (T_3 - T_4) + (T_4 - T_c) = \dot{Q}_x (R_h + R_1 + R_2 + R_3 + R_c)$$

Each intermediate temperature appears once with a positive sign and once with a negative sign. These terms cancel, leaving the overall temperature difference $T_h - T_c$. Therefore,

$$\dot{Q}_x = \frac{T_h - T_c}{\sum R} = U A (T_h - T_c) \tag{2}$$

The overall heat transfer coefficient is

$$U = \frac{1}{A \sum R}$$

Here

$$\sum R = R_h + R_1 + R_2 + R_3 + R_c.$$

The overall heat transfer coefficient U, in $\mathrm{W\ m^{-2}\ K^{-1}}$, combines the effects of both convection processes and all three solid layers.

We can compare this wall to an electrical circuit with resistors in series. A temperature difference drives heat transfer, just as a voltage difference drives electric current. The same heat rate passes through both fluid films and all three layers, so their thermal resistances add to give the total resistance.

Once we know the heat rate, we can calculate the surface and interface temperatures. Starting at the hot fluid, subtract the temperature drop across each resistance in turn. This gives $T_1 = T_h - \dot{Q}_x R_h$ at the hot wall surface, followed by $T_2 = T_1 - \dot{Q}_x R_1$ at the first interface. Since each drop is $\dot{Q}_x R$ and the heat rate is the same throughout, the largest temperature drop occurs across the largest resistance.

At fixed fluid temperatures, reducing the total resistance increases the heat rate. At fixed area, the film resistance is inversely proportional to the heat transfer coefficient h. Increasing the fluid speed generally increases h and therefore reduces the film resistance. A thinner solid

layer or a material with higher conductivity reduces the conduction resistance. The effect depends on how much that resistance contributes to the total. Halving a small resistance changes the total much less than halving the largest one. For insulation, the aim is to increase the total resistance and reduce heat loss.

4.2 Series and parallel composite wall

Let's consider the wall in Figure 3. Layers E and H extend across the full area A. Between them, materials F and G each occupy half the area and have the same thickness, $L_F = L_G$. The outer faces are held at uniform temperatures T_1 and T_2, with $T_1 > T_2$.

Assume steady conduction, constant thermal conductivities, no heat generation and perfect contact between materials. The top and bottom edges are insulated.

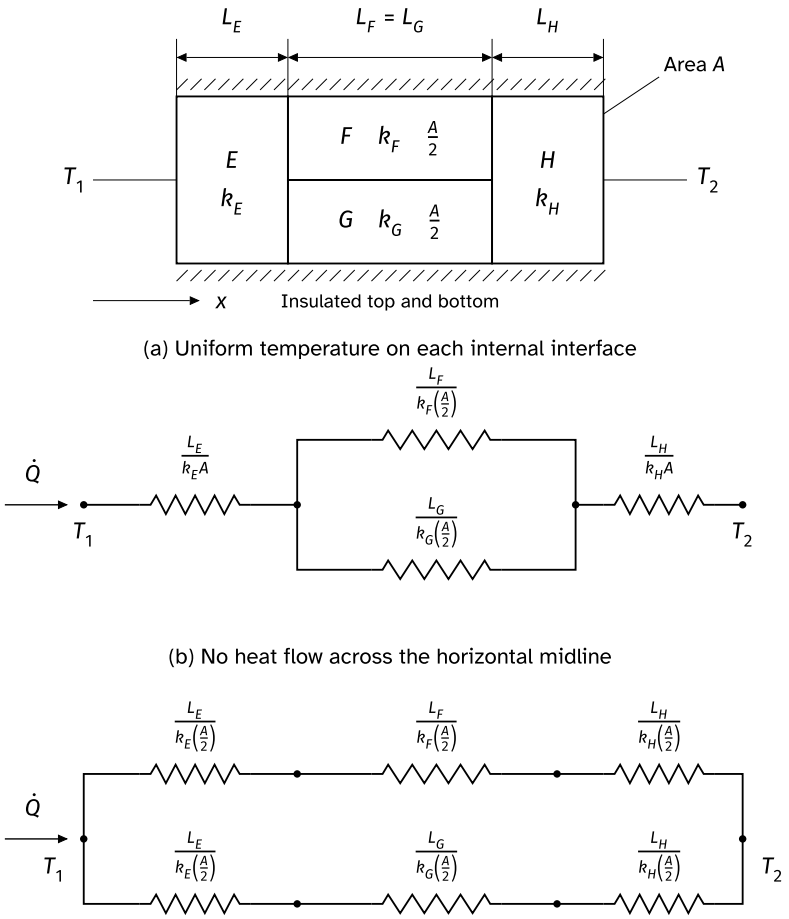


Figure 3: Composite wall with thicknesses, conductivities and areas labelled. Network (a) assumes uniform temperatures on the internal interfaces; network (b) prevents heat flow across the horizontal midline.

Heat passes through E, then through F or G, and finally through H. To represent this wall by a resistance network, we need an assumption about how temperature varies across the internal interfaces. The figure shows two possible models.

In model (a), we assume that each internal interface has a uniform temperature. F and G therefore have the same temperature difference ΔT across them. Their heat rates depend on their individual resistances and add to give the total heat rate:

$$\dot{Q} = \dot{Q}_F + \dot{Q}_G = \frac{\Delta T}{R_F} + \frac{\Delta T}{R_G} = \Delta T\left(\frac{1}{R_F} + \frac{1}{R_G}\right)$$

F and G are in parallel. We can replace them with an equivalent resistance R_{FG} such that $\dot{Q} = \Delta T / R_{FG}$. Comparing this expression with the equation above gives

$$\frac{1}{R_{FG}} = \frac{1}{R_F} + \frac{1}{R_G}, \qquad R_{FG} = \frac{R_F R_G}{R_F + R_G} \tag{3}$$

Each resistance must use the area through which heat passes. E and H occupy the full area, whereas F and G each occupy $A/2$:

$$R_E = \frac{L_E}{k_E A}, \qquad R_H = \frac{L_H}{k_H A}$$

$$R_F = \frac{L_F}{k_F\left(\frac{A}{2}\right)}, \qquad R_G = \frac{L_G}{k_G\left(\frac{A}{2}\right)}$$

The total heat rate passes through E and H, so these two resistances are in series with the parallel combination of F and G. The equivalent resistance for model (a) is therefore

$$R_a = R_E + R_{FG} + R_H$$

In model (b), we instead assume that no heat crosses the horizontal midline. Heat then follows two separate paths through the wall. The upper path passes through E, F and H; the lower path passes through E, G and H. Each path has three resistances in series, and the two complete paths are in parallel between T_1 and T_2.

Each path uses half the area of E and H. Since conduction resistance is inversely proportional to area, their resistances in each path are $2R_E$ and $2R_H$. The definitions of R_F and R_G already use $A/2$. The equivalent resistance for model (b) is therefore

$$\frac{1}{R_b} = \frac{1}{2R_E + R_F + 2R_H} + \frac{1}{2R_E + R_G + 2R_H} \tag{4}$$

The actual wall generally has two-dimensional heat flow. Heat can pass between its upper and lower parts, and the internal interface temperatures need not be uniform. Under the stated assumptions, the two models give lower and upper bounds on its thermal resistance:

$$R_a \le R \le R_b$$

If F and G have the same conductivity, the two models give the same result because the temperature varies only through the wall thickness. If the estimates differ substantially, a two-dimensional conduction calculation is needed to determine the resistance more precisely.

For either model, use its equivalent resistance to calculate the total heat rate from

$$\dot{Q} = \frac{T_1 - T_2}{R}.$$

Dividing this heat rate by the full area A gives the average heat flux through the wall.

4.3 Thermal contact resistance

So far, we have assumed that adjacent layers are in perfect thermal contact, with no temperature difference at their interface. Real surfaces have small irregularities. When two solids are pressed together, contact occurs at a limited number of points, leaving gaps elsewhere, as shown in Figure 4. Heat must pass through these contact regions and any material in the gaps, adding a thermal resistance at the interface.

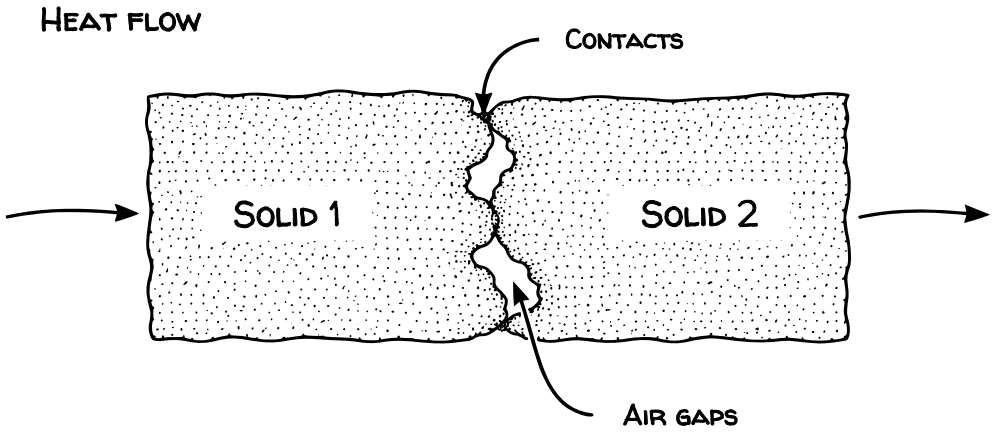


Figure 4: Two rough solid surfaces touch at small contact areas, with air gaps between them. Heat flows from the left solid to the right solid.

We describe heat transfer across the interface using the thermal contact conductance h_{contact}. Let T_- and T_+ be the temperatures of the two contacting surfaces. The heat rate and contact resistance are

$$\dot{Q} = h_{\text{contact}} A (T_- - T_+)$$

The corresponding contact resistance is

$$R_{\text{contact}} = \frac{T_- - T_+}{\dot{Q}} = \frac{1}{h_{\text{contact}} A}$$

Here A is the nominal contact area, rather than the sum of the small areas where the solids actually touch. The contact conductance has units of $\mathrm{W\,m^{-2}\,K^{-1}}$, and the contact resistance has units of $\mathrm{K\,W^{-1}}$. We add this resistance to the other resistances in series. The resulting temperature drop at the interface is $T_- - T_+ = \dot{Q} R_{\text{contact}}$.

QUIZ

3. Series and parallel networks

A cold-room panel has two continuous linings with insulation between them. A solid metal support passes through part of the insulation, connecting the two linings. Treat each surface between a lining and the core as having a uniform temperature.

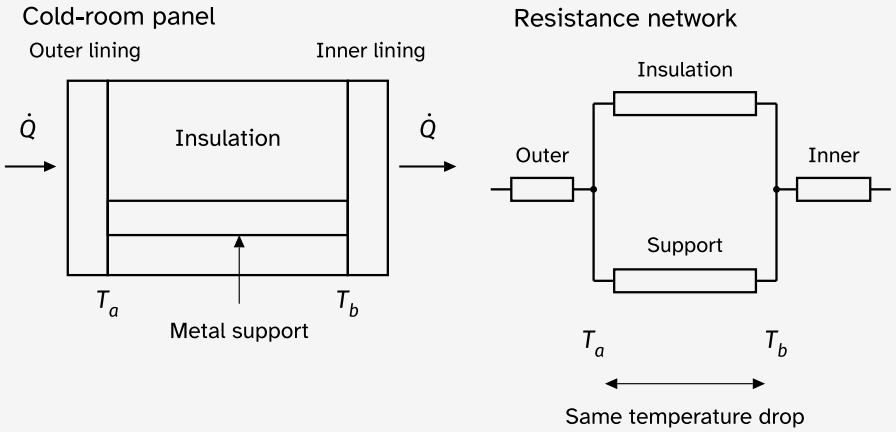


Figure 5: Heat passes through an outer lining, splits between insulation and a metal support, then passes through the inner lining. The two core paths connect the same interface temperatures.

Heat flows steadily from the warmer surroundings into the cold room, with no heat generation in the panel. Which statement must be true?

1. The two linings have equal temperature drops, and the insulation and support have equal heat rates.
2. The same heat rate passes through each lining, the insulation and the support.
3. Both linings, the insulation and the support all have the same temperature drop.
4. The same total heat rate passes through each lining. The insulation and metal support have the same temperature drop.

See solution for Quiz 3

QUIZ

4. The largest temperature drop

An engineer is checking the temperature profile through a furnace wall. For a $1\ \mathrm{m}^2$ section, the thermal resistances from the hot gas to the room air are listed below. Assume steady heat transfer with no heat generation. All five resistances are in series.

Part	Thermal resistance R in $\mathrm{K\,W^{-1}}$
Hot-side gas film	0.040
Firebrick	0.100
Insulation	1.200
Steel casing	0.000125
Cold-side air film	0.100

Across which part should the engineer expect the largest temperature drop?

1. The insulation layer.
2. The steel casing, because steel has a high thermal conductivity.
3. The hot-side gas film, because it is closest to the furnace.
4. All five parts should have the same temperature drop.

See solution for Quiz 4

4.4 Series of cylinders

Let's apply the resistance method to the wall of a long pipe. The wall has length L, inner radius r_1, outer radius r_2 and thermal conductivity k. Unlike a plane wall, its area normal to heat flow changes through the thickness.

Assume steady conduction, constant conductivity, no heat generation and negligible heat loss through the ends. The surface conditions are uniform along the pipe and around its circumference. Temperature therefore varies only with radius, $T = T(r)$.

At steady state, heat enters and leaves each part of the wall at the same rate. The heat rate $\dot{Q}_r$ is therefore independent of radius. At radius r, heat passes through the cylindrical area $2\pi r L$, so Fourier's law gives

$$\dot{Q}_r = -2\pi r k L \frac{dT}{dr}$$

Let the inner and outer surface temperatures be $T_1 = T(r_1)$ and $T_2 = T(r_2)$. Separating the temperature and radius terms, then integrating between these surfaces, gives

$$\dot{Q}_r \int_{r_1}^{r_2} \frac{dr}{r} = -2\pi k L \int_{T_1}^{T_2} dT$$

Evaluating the integrals gives

$$\dot{Q}_r \ln\left(\frac{r_2}{r_1}\right) = 2\pi k L (T_1 - T_2)$$

We can now write the result as

$$\dot{Q}_r = \frac{T_1 - T_2}{R}.$$

Comparing this form with the integrated equation gives the cylindrical resistance:

$$R_{\text{cyl}} = \frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi k L} \tag{5}$$

The heat rate is then

$$\dot{Q}_r = \frac{T_1 - T_2}{R_{\text{cyl}}}$$

Integrating between r_1 and any radius r within the wall gives

$$T(r) = T_1 - (T_1 - T_2)\frac{\ln\left(\frac{r}{r_1}\right)}{\ln\left(\frac{r_2}{r_1}\right)}$$

Temperature varies logarithmically with radius because the area changes through the wall. It therefore does not have the straight-line profile obtained for a plane wall with constant conductivity.

The local heat flux is

$$q_r = \frac{\dot{Q}_r}{2\pi r L}.$$

For outward heat flow, the same heat rate passes through progressively larger areas. The heat flux therefore decreases as $1/r$, even though the heat rate remains constant.

Now consider three concentric layers, A, B and C, with successive radii r_1, r_2, r_3, r_4. Heat passes from the inner fluid through its film, through all three layers, and then through the outer film to the outer fluid. These five resistances are in series, as shown in Figure 6.

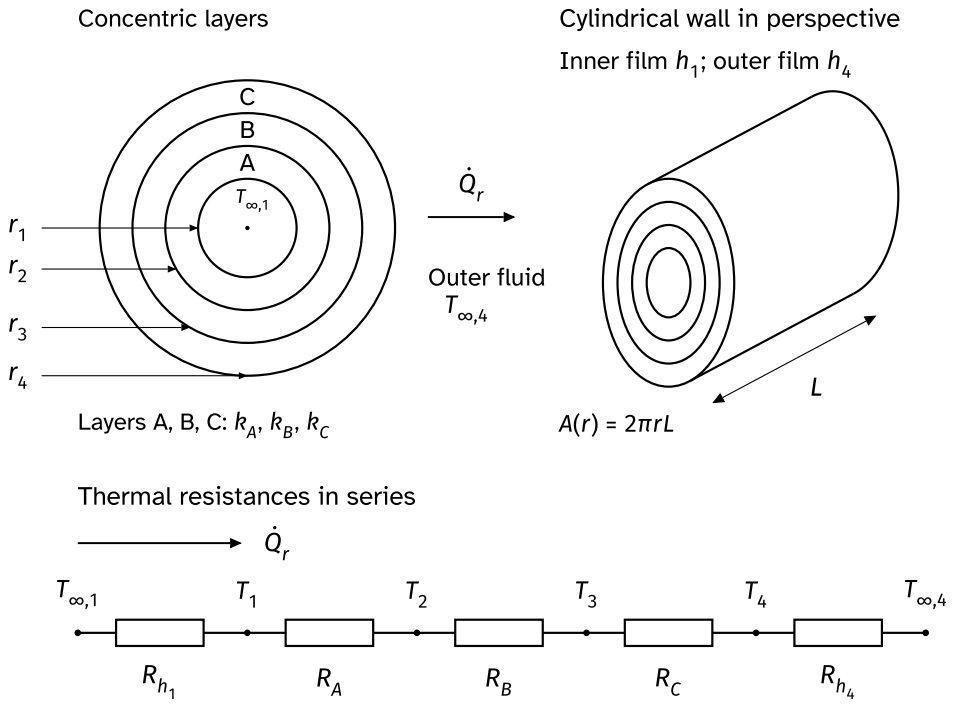


Figure 6: Layered cylindrical wall: cross-section, pipe geometry and series resistance network. The same heat rate passes through both films and all three layers.

The total resistance is

$$R_{\text{total}} = \frac{1}{h_1 2\pi r_1 L} + \frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi k_A L} + \frac{\ln\left(\frac{r_3}{r_2}\right)}{2\pi k_B L} + \frac{\ln\left(\frac{r_4}{r_3}\right)}{2\pi k_C L} + \frac{1}{h_4 2\pi r_4 L} \tag{6}$$

The overall temperature difference is between the two fluids. With inner and outer fluid temperatures $T_{\infty,1}$ and $T_{\infty,4}$, the heat rate is

$$\dot{Q}_r = \frac{T_{\infty,1} - T_{\infty,4}}{R_{\text{total}}}.$$

The film resistances use different areas: $A_i = 2\pi r_1 L$ on the inside and $A_o = 2\pi r_4 L$ on the outside. We must therefore state which area is used when defining an overall heat transfer coefficient. Either choice gives the same heat rate, provided that

$$U_i A_i = U_o A_o = \frac{1}{R_{\text{total}}}.$$

The values of U_i and U_o themselves are generally different.

For a thin wall, $r_2 - r_1 \ll r_1$, the change in area is small. Using

$$\ln\left(\frac{r_2}{r_1}\right) \simeq \frac{r_2 - r_1}{r_1}$$

in Eq. 5 gives the plane-wall expression with thickness $r_2 - r_1$ and area $2\pi r_1 L$.

$$R_{\text{cyl}} \simeq \frac{r_2 - r_1}{2\pi r_1 k L} = \frac{\Delta x}{kA}$$

Here $\Delta x = r_2 - r_1$ and $A = 2\pi r_1 L$.

4.5 Series of spheres

The same resistance method applies to a hollow spherical wall, such as insulation around a spherical process vessel. Let its inner and outer radii be r_1 and r_2, and its thermal conductivity be k.

Assume steady conduction, constant conductivity and no heat generation. The surface conditions are uniform over each spherical surface, so temperature varies only with radius, $T = T(r)$.

At steady state, the same heat rate passes through every spherical surface. At radius r, the area normal to heat flow is $4\pi r^2$. Fourier's law therefore gives

$$\dot{Q}_r = -4\pi r^2 k \frac{dT}{dr}$$

Let the inner and outer surface temperatures be $T_1 = T(r_1)$ and $T_2 = T(r_2)$. Separating the temperature and radius terms, then integrating between the surfaces, gives

$$\dot{Q}_r \int_{r_1}^{r_2} \frac{dr}{r^2} = -4\pi k \int_{T_1}^{T_2} dT$$

Evaluating the integrals gives

$$\dot{Q}_r \left(\frac{1}{r_1} - \frac{1}{r_2} \right) = 4\pi k (T_1 - T_2)$$

We can now write the result as

$$\dot{Q}_r = \frac{T_1 - T_2}{R}.$$

Comparing this form with the integrated equation gives the spherical resistance:

$$R_{\text{sph}} = \frac{\frac{1}{r_1} - \frac{1}{r_2}}{4\pi k} \tag{7}$$

The heat rate is then

$$\dot{Q}_r = \frac{T_1 - T_2}{R_{\text{sph}}}$$

Integrating between r_1 and any radius r within the wall gives

$$T(r) = T_1 - (T_1 - T_2) \frac{\frac{1}{r_1} - \frac{1}{r}}{\frac{1}{r_1} - \frac{1}{r_2}}$$

Temperature varies linearly with $1/r$, rather than with r. The local heat flux is

$$q_r = \frac{\dot{Q}_r}{4\pi r^2}.$$

For outward heat flow, it decreases as $1/r^2$ because the same heat rate passes through progressively larger areas.

Now consider three concentric spherical layers, A, B and C, with radii r_1, r_2, r_3, r_4. Heat passes through the inner fluid film, all three solid layers and the outer fluid film. Their resistances add in series, as shown in Figure 7.

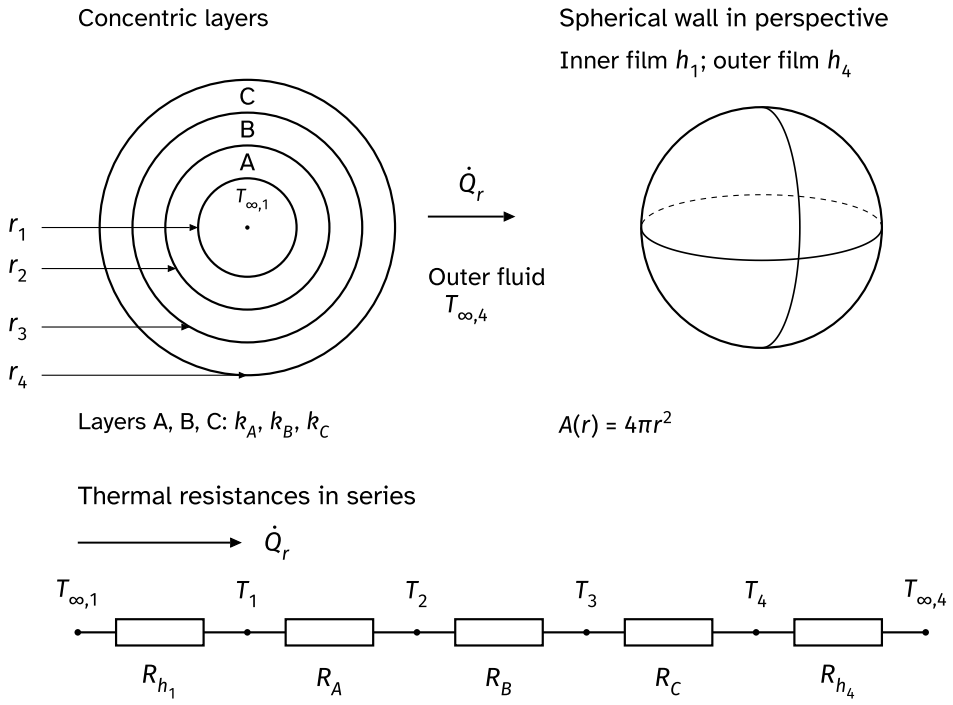


Figure 7: Layered spherical wall: central section, spherical geometry and series resistance network. Each layer has a different area as the heat flows outwards.

The total resistance is

$$R_{\text{total}} = \frac{1}{h_1 4\pi r_1^2} + \frac{\frac{1}{r_1} - \frac{1}{r_2}}{4\pi k_A} + \frac{\frac{1}{r_2} - \frac{1}{r_3}}{4\pi k_B} + \frac{\frac{1}{r_3} - \frac{1}{r_4}}{4\pi k_C} + \frac{1}{h_4 4\pi r_4^2} \tag{8}$$

With inner and outer fluid temperatures $T_{\infty,1}$ and $T_{\infty,4}$, the heat rate is

$$\dot{Q}_r = \frac{T_{\infty,1} - T_{\infty,4}}{R_{\text{total}}}$$

Each film uses the area of its own surface, $A_i = 4\pi r_1^2$ or $A_o = 4\pi r_4^2$. Either area can define an overall heat transfer coefficient, provided that

$$U_i A_i = U_o A_o = \frac{1}{R_{\text{total}}}.$$

For a thin spherical wall, $r_2 - r_1 \ll r_1$, the two surface areas are nearly equal. Writing

$$\frac{1}{r_1} - \frac{1}{r_2} = \frac{r_2 - r_1}{r_1 r_2}$$

in Eq. 7 and taking $r_2 \simeq r_1$ gives the plane-wall limit:

$$R_{\text{sph}} \simeq \frac{r_2 - r_1}{4\pi k r_1^2} = \frac{\Delta x}{kA}$$

Here $\Delta x = r_2 - r_1$ and $A = 4\pi r_1^2$.

EXERCISE

Exercise 1. Insulation around a spherical vessel

A spherical vessel has insulation extending from $r_1 = 0.50\ \text{m}$ to $r_2 = 0.55\ \text{m}$. Take the insulation conductivity as $k = 0.040\ \text{W m}^{-1}\ \text{K}^{-1}$. Its inner and outer insulation surface temperatures are $80\,^\circ\text{C}$ and $20\,^\circ\text{C}$. Assume steady radial conduction and neglect heat flow through supports and pipe connections. What is the heat loss through the insulation?

See solution for Exercise 1

4.5.1 Conduction into an infinite surrounding fluid

Consider a sphere surrounded by an infinite, stationary fluid, with heat transfer through the fluid by conduction alone. Taking $r_2 \to \infty$ in Eq. 7 gives

$$R = \frac{1}{4\pi k r_1},$$

where k is the fluid's thermal conductivity. Equating this to

$$\frac{1}{h(4\pi r_1^2)}$$

gives $h = k/r_1$.

Conduction limit for a sphere

For a sphere in an infinite, stationary fluid, pure conduction gives

$$Nu_D = \frac{hD}{k_{\text{fluid}}} = 2$$

Here Nu_D is the dimensionless Nusselt number based on the sphere diameter $D = 2r_1$, and k_{fluid} is the fluid's thermal conductivity. Heat transfer persists even without bulk fluid motion. This is the conduction limit approached by sphere convection correlations as fluid motion vanishes.

QUIZ

5. Radial heat rate and flux

A long steam pipe is insulated. Heat leaks outwards through the insulation at steady state. Temperature varies only with radius, and there is no heat generation in the insulation or heat loss through the ends.

Moving from the pipe surface to the outside of the insulation, what happens to the total heat rate $\dot{Q}_r$ and local heat flux q_r?

1. Both the heat rate and heat flux increase.
2. The heat rate decreases, while the heat flux stays the same.
3. The heat rate stays the same, while the heat flux decreases.
4. Both the heat rate and heat flux stay the same.

See solution for Quiz 5

4.6 Classroom example 1: glass window

EXERCISE

Exercise 2. Glass window: heat loss and inside surface temperature

Consider a 0.8-m-high and 1.5-m-wide glass window with a thickness of 8 mm and a thermal conductivity of $k = 0.78\ \mathrm{W/(m \cdot {}^\circ C)}$. Determine the steady rate of heat transfer through this glass window and the temperature of its inner surface for a day during which the room is maintained at $20\ ^\circ\mathrm{C}$ while the temperature of the outdoors is $-10\ ^\circ\mathrm{C}$. Take the heat transfer coefficients on the inner and outer surfaces of the window to be $h_1 = 10\ \mathrm{W/(m^2 \cdot {}^\circ C)}$ and $h_2 = 40\ \mathrm{W/(m^2 \cdot {}^\circ C)}$, which includes the effects of radiation.

See solution for Exercise 2

5 Case 2: 1D steady conduction with heat generation

In the previous examples, no heat was generated within the wall. We now consider a solid with internal heat generation. At steady state, each region must lose both the heat entering it and the heat generated within it. We will use this balance to find the temperature distribution in an electrically heated wire.

5.1 Electric wire and shell energy balance

Consider a long wire of radius R and length L. A uniform electrical current density I passes through it. Here I is current per unit area, in $\mathrm{A\,m^{-2}}$. Let k_e be the electrical conductivity, in $\mathrm{S\,m^{-1}}$, and k the thermal conductivity. Electrical resistance produces heat throughout the wire at the volumetric rate $S_e = I^2/k_e$, in $\mathrm{W\,m^{-3}}$.

Assume steady state, constant thermal and electrical conductivities, uniform heat generation and negligible heat loss through the ends. The wire is stationary, and its surface is held at a uniform temperature T_0. Conditions do not vary around the circumference.

Temperature then depends only on radius, $T = T(r)$. This makes the conduction problem one-dimensional, although the wire itself has three dimensions. Variation along the wire or around its circumference would require a model with additional spatial coordinates.

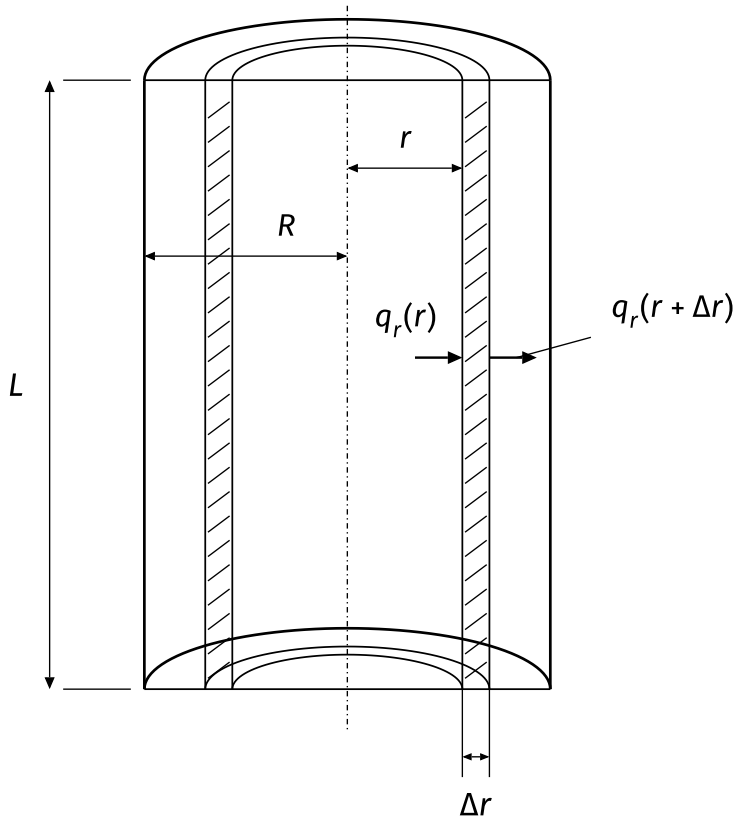


Figure 9: Longitudinal cutaway of a solid wire of radius R and length L. The narrow hatched strips are cuts through the annular control volume, with inner radius r and thickness delta r. Arrows show the radial heat flux at its inner and outer faces.

The shell balance follows Shah (n.d.), slides 22 and 23, and Bird et al. (2006), section 10.2.

Choose the thin cylindrical shell between r and $r + \Delta r$ in Figure 9. Heat enters its inner face at the rate $2\pi r L q_r(r)$ and leaves its outer face at the rate

$$2\pi(r + \Delta r)Lq_r(r + \Delta r).$$

The areas differ because the two faces are at different radii.

The shell volume is $\pi[(r + \Delta r)^2 - r^2]L$. Multiplying it by S_e gives the heat generated within the shell. With no accumulation, the balance is heat in minus heat out plus heat generated equals zero:

$$0 = 2\pi r L q_r(r) - 2\pi(r + \Delta r)Lq_r(r + \Delta r) + S_e\pi[(r + \Delta r)^2 - r^2]L \tag{9}$$

Dividing this balance by $2\pi L\Delta r$ and moving the heat-flow terms to the left gives

$$\frac{(r + \Delta r)q_r(r + \Delta r) - rq_r(r)}{\Delta r} = S_e\left(r + \frac{\Delta r}{2}\right)$$

The left side is the change in rq_r divided by the shell thickness. Taking the limit as $\Delta r \to 0$ gives

$$\frac{d(rq_r)}{dr} = rS_e \tag{10}$$

The radius remains inside the derivative because the heat-flow area changes with r. Treating the two shell faces as having the same area would give an incorrect radial balance. In a finite-volume calculation, we can instead apply Eq. 9 to each cell, retaining its actual inner and outer face areas.

5.2 Integrate and apply the boundary conditions

To find the heat flux, integrate Eq. 10 with respect to radius:

$$rq_r = \frac{S_e r^2}{2} + C_1 \qquad q_r = \frac{S_e r}{2} + \frac{C_1}{r}$$

At the axis, the flux must be finite. A nonzero C_1 would make C_1/r unbounded as $r \to 0$, so $C_1 = 0$. The resulting flux, $q_r = S_e r/2$, is zero at the axis, consistent with symmetry.

Substituting $q_r = -k\,dT/dr$ gives the temperature gradient. A second integration gives the temperature:

$$\frac{dT}{dr} = -\frac{S_e r}{2k} \qquad T = -\frac{S_e r^2}{4k} + C_2$$

We determine C_2 from the prescribed surface temperature $T(R) = T_0$. Substitution gives

$$C_2 = T_0 + \frac{S_e R^2}{4k},$$

and hence

$$T(r) - T_0 = \frac{S_e}{4k}(R^2 - r^2) \tag{11}$$

The radial heat flux is

$$q_r = \frac{S_e r}{2}$$

Temperature decreases parabolically from its maximum at the axis to T_0 at the surface. The radial heat flux behaves differently: it is zero at the axis and increases linearly with radius. Each cylindrical surface must release all the heat generated inside it. At the axis, the temperature gradient is zero, so Fourier's law gives zero flux even though the temperature is highest there.

The centre-to-surface temperature difference is $S_e R^2/(4k)$. Doubling R therefore increases this difference by a factor of four when S_e and k are fixed. The condition of fixed S_e matters. If the total electrical current were held fixed instead, changing the radius would also change the current density and the heat generation per unit volume.

QUIZ

6. Centreline symmetry

A long solid wire generates heat uniformly, and its surface is held at T_0. At steady state, temperature varies only with radius. What must be true at the axis, $r = 0$?

1. The temperature equals the surface temperature T_0.
2. The temperature gradient dT/dr and radial heat flux are both zero.
3. The radial heat flux is infinite because the cylindrical area is zero.
4. The temperature gradient has its largest magnitude.

See solution for Quiz 6

QUIZ

7. Doubling the wire radius

A long wire generates heat uniformly at a rate S_e per unit volume. At steady state, its centre is $12\ \mathrm{K}$ hotter than its surface. Suppose we increase the radius from $2\ \mathrm{mm}$ to $4\ \mathrm{mm}$, keeping S_e, thermal conductivity k and surface temperature T_0 unchanged. Neglect heat loss through the ends. What is the new centre-to-surface temperature difference?

1. $6\ \mathrm{K}$.
2. $12\ \mathrm{K}$.
3. $24\ \mathrm{K}$.
4. $48\ \mathrm{K}$.

See solution for Quiz 7

5.3 Average temperature and conservation

To find the mean temperature, we must weight each temperature by the volume of material at that radius. A ring of thickness dr has area $2\pi r\,dr$, so equal radial intervals contain more material near the surface than near the axis.

Temperature is uniform along the wire, so the length cancels from the volume average. We can therefore average over the cross-section. Using Eq. 11 gives

$$\overline{T} - T_0 = \frac{\int_0^R \left[\frac{S_e(R^2 - r^2)}{4k}\right] 2\pi r\,dr}{\pi R^2} = \frac{S_e R^2}{8k} \tag{12}$$

Evaluating the integrals uses

$$\int_0^R r\,dr = \frac{R^2}{2}$$

and

$$\int_0^R (R^2 r - r^3)\,dr = \frac{R^4}{4}.$$

The mean temperature rise is half the centre-to-surface rise. Averaging temperatures at equally spaced radii without the factor r would overestimate the mean by giving too much weight to the hotter material near the axis.

We can check conservation of energy using the flux at $r = R$. Multiplying by the curved surface area gives the total heat loss:

$$\dot{Q}_R = q_r(R)\, 2\pi RL = \frac{S_e R}{2}\, 2\pi RL = S_e \pi R^2 L$$

The result is exactly the heat generation per unit volume multiplied by the wire volume. All the heat generated leaves through the curved surface, as required by the assumptions. A numerical solution should satisfy this overall balance as well as the local energy equation.

EXERCISE

Exercise 3. Temperature and conservation in a heated wire

A wire has radius $R = 0.002\,\mathrm{m}$, length $L = 0.50\,\mathrm{m}$, thermal conductivity $k = 15\,\mathrm{W\,m^{-1}\,K^{-1}}$ and electrical conductivity $k_e = 10^6\,\mathrm{S\,m^{-1}}$. Its current density is $I = 10^6\,\mathrm{A\,m^{-2}}$, and its surface is held at $T_0 = 25\,^\circ\mathrm{C}$. Use the wire model above to calculate the centre temperature, mean temperature and heat loss through the curved surface. Verify that the heat loss equals the total heat generation.

See solution for Exercise 3

6 Reducing the general energy equation

The shell balance gave the energy equation for a particular geometry. We can also obtain it by simplifying the general energy equation. For an incompressible Newtonian fluid with constant properties and negligible compression work, this equation can be written as

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T \right) = k \nabla^2 T + S \tag{13}$$

Here ρ is density, $\hat{C}_p$ is specific heat capacity per unit mass and $\boldsymbol{u}$ is fluid velocity. The source S represents heat generation per unit volume, in $\mathrm{W\,m^{-3}}$. Viscous dissipation must either be negligible or be included in S.

The left side accounts for the local temperature change and heat transport by fluid motion. The terms on the right account for conduction and internal heat generation. In the stationary wire, $\boldsymbol{u} = 0$ and $S = S_e$.

6.1 Cartesian and cylindrical forms

The material derivative DT/Dt describes the rate of temperature change of a fluid element as it moves. It combines the local time derivative with the change due to motion through a temperature field. In Cartesian coordinates, the material derivative is

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u_x \frac{\partial T}{\partial x} + u_y \frac{\partial T}{\partial y} + u_z \frac{\partial T}{\partial z}$$

The Laplacian is

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

For cylindrical coordinates, we use radius r, angle θ and axial position z. The distances are measured in metres and the angle in radians. The material derivative is

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u_r \frac{\partial T}{\partial r} + \frac{u_\theta}{r}\frac{\partial T}{\partial \theta} + u_z \frac{\partial T}{\partial z} \tag{14}$$

The Laplacian is

$$\nabla^2 T = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \tag{15}$$

Substituting these expressions into Eq. 13 gives its cylindrical form. The radial conduction term contains the same factor r as the shell balance because the area normal to radial heat flow increases with radius.

6.2 Thermal diffusivity

DEFINITION
Thermal diffusivity

Thermal diffusivity, α, is the ratio of thermal conductivity to volumetric heat capacity, $\alpha = k/(\rho \hat{C}_p)$. It characterises the rate at which temperature disturbances spread through a material by conduction. Its SI unit is $\mathrm{m^2\,s^{-1}}$.

The volumetric heat capacity $\rho \hat{C}_p$ is the energy required per unit volume for a unit temperature rise. Dividing Eq. 13 by this quantity gives

$$\frac{DT}{Dt} = \alpha \nabla^2 T + \frac{S}{\rho \hat{C}_p}$$

Here the thermal diffusivity is

$$\alpha = \frac{k}{\rho \hat{C}_p} \tag{16}$$

Each term in the temperature equation has units of $\mathrm{K\,s^{-1}}$. Thermal diffusivity combines two effects. A higher conductivity allows heat to redistribute more readily, while a higher volumetric heat capacity requires more energy for the same temperature change. Their ratio determines the rate at which temperature differences diminish by conduction.

6.3 Example: estimating a diffusion time

We can use thermal diffusivity to estimate the time required for conduction over a distance L. With no fluid motion or heat generation, the energy equation is

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T.$$

For a temperature change of order ΔT over distance L and time t, the two terms have the approximate magnitudes

$$\frac{\partial T}{\partial t} \sim \frac{\Delta T}{t} \qquad \alpha \frac{\partial^2 T}{\partial y^2} \sim \alpha \frac{\Delta T}{L^2}$$

Balancing these magnitudes and cancelling ΔT gives

$$t_{\text{diff}} \sim \frac{L^2}{\alpha} \tag{17}$$

The diffusion time scale therefore increases with the square of the distance. For the same material, doubling L increases the estimated time by a factor of four.

For example, consider a piece of beef $2\,\text{cm}$ thick, heated from both sides. Conduction must act over $L = 0.01\ \text{m}$ to reach the centre. Taking $\alpha = 1.4 \times 10^{-7}\ \text{m}^2\,\text{s}^{-1}$ gives $t_{\text{diff}} \sim 714\,\text{s}$, or about twelve minutes.

This is a characteristic time for temperature changes to extend through the thickness. Finding the time for the centre to reach a specified temperature requires the initial and boundary conditions. The estimate also does not predict surface browning, which involves chemical reactions.

QUIZ

8. Conductivity and diffusivity

A mould is repeatedly heated and cooled. A designer proposes using an alloy with a higher thermal conductivity so that temperature changes spread through the mould faster. Is conductivity alone enough to tell whether the proposed alloy has a higher thermal diffusivity?

1. Yes. Thermal diffusivity depends only on thermal conductivity.
2. Yes. Density and heat capacity affect steady conduction only.
3. No. We also need the density and specific heat capacity of each alloy.
4. No. A higher conductivity always gives a lower diffusivity.

See solution for Quiz 8

QUIZ

9. Diffusion distance and time

In a moulding process, a polymer sheet is heated from both sides. For a $10\ \mathrm{mm}$ thick sheet, the estimated diffusion time from either surface to the centre is $50\ \mathrm{s}$.

What is the estimated diffusion time for a $30\ \mathrm{mm}$ thick sheet of the same polymer? Assume the thermal diffusivity is unchanged and use $t_{\mathrm{diff}} \sim L^2/\alpha$, where L is the distance from a heated surface to the centre.

1. $150\ \mathrm{s}$.
2. $450\ \mathrm{s}$.
3. $50\ \mathrm{s}$.
4. $50/9\ \mathrm{s}$.

See solution for Quiz 9

6.4 Apply the general equation to the wire

Let's recover the wire equation from the general energy equation. Substitute the material derivative in Eq. 14 and the Laplacian in Eq. 15 into Eq. 13, then apply the wire assumptions:

1. At steady state, $\partial T/\partial t = 0$.
2. The wire is stationary, so $u_r = u_\theta = u_z = 0$ and there is no heat transport by bulk motion.
3. Conditions are uniform around the circumference, so derivatives with respect to θ are zero.
4. End effects are negligible, so derivatives with respect to z are zero in the region considered.

Only radial conduction and heat generation remain. Taking $S = S_e$ and constant k gives

$$0 = \frac{k}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + S_e \tag{18}$$

Rearranging after multiplication by r/k, then integrating, gives

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = -\frac{S_e r}{k} \qquad \frac{dT}{dr} = -\frac{S_e r}{2k} + \frac{C}{r}$$

A nonzero C would produce a logarithmic temperature term that diverges at $r = 0$. The temperature must remain finite, so $C = 0$. A second integration, followed by the boundary condition $T(R) = T_0$, recovers Eq. 11.

We can also compare the differential equations. Substituting Fourier's law, $q_r = -k\,dT/dr$, into Eq. 18 gives

$$\frac{d(rq_r)}{dr} = rS_e,$$

exactly as in Eq. 10. The shell balance and the general energy equation therefore give the same model. The first derives it from heat entering, leaving and being generated in a control volume; the second obtains it by simplifying an existing energy equation.

QUIZ

10. Reducing the energy equation

We want the steady temperature distribution in a long, stationary wire that generates heat uniformly. Temperature does not vary along the wire or around its circumference, and heat loss through the ends is negligible. Which terms remain in the energy equation after applying these assumptions?

1. Accumulation and radial conduction.
2. Advection and internal heat generation.
3. Radial and axial conduction.
4. Radial conduction and internal heat generation.

See solution for Quiz 10

7 Additional examples

EXERCISE

Exercise 4. Temperature drops through a furnace wall

A furnace wall separates hot gas at $600\,^{\circ}\mathrm{C}$ from air at $25\,^{\circ}\mathrm{C}$. From the hot side, the layers are $0.12\ \mathrm{m}$ of firebrick, $0.060\ \mathrm{m}$ of insulation and $0.0020\ \mathrm{m}$ of steel. Their thermal conductivities are 1.2, 0.050 and $16\ \mathrm{W\,m^{-1}\,K^{-1}}$, respectively. The hot-side and cold-side heat transfer coefficients are 25 and $10\ \mathrm{W\,m^{-2}\,K^{-1}}$, respectively.

Assume steady, one-dimensional heat transfer, constant properties, perfect contact and negligible radiation. Calculate the heat flux and all surface and interface temperatures. Determine which solid layer has the largest temperature drop. Then calculate the heat flux if the insulation thickness is doubled, with all other conditions unchanged.

See solution for Exercise 4

EXERCISE

Exercise 5. Insulation and boil-off from a spherical vessel

A thin spherical vessel of radius $0.50\ \mathrm{m}$ holds boiling liquid nitrogen at $77\ \mathrm{K}$. Its shell is at the liquid temperature, and the surrounding air is at $293\ \mathrm{K}$. Insulation with conductivity $0.040\ \mathrm{W\,m^{-1}\,K^{-1}}$ extends from the vessel surface to an outer radius of $0.54\ \mathrm{m}$.

Take the outside heat transfer coefficient as $10\ \mathrm{W\,m^{-2}\,K^{-1}}$ for both insulated and bare vessels. Neglect shell resistance, radiation and heat transfer through supports. The liquid remains at its boiling temperature.

Calculate the heat leak into the insulated vessel, the nitrogen boil-off rate and the outer insulation temperature. Compare the heat leak and boil-off rate with those of the bare

vessel. The latent heat is $198\ \mathrm{kJ\,kg^{-1}}$. Use the supplied outside coefficient; no convection correlation is required.

See solution for Exercise 5

EXERCISE

Exercise 6. A heat-generating cylinder with an unknown surface temperature

A long solid cylinder has radius $0.010\ \mathrm{m}$, length $1.0\ \mathrm{m}$ and thermal conductivity $5.0\ \mathrm{W\,m^{-1}\,K^{-1}}$. Uniform heat generation occurs at $S_e = 2.0 \times 10^6\ \mathrm{W\,m^{-3}}$. Its curved surface is cooled by fluid at 300 K, with a heat transfer coefficient $h = 200\ \mathrm{W\,m^{-2}\,K^{-1}}$.

Assume steady radial conduction, constant properties and negligible heat loss through the ends. Derive the temperature profile and calculate the surface and centre temperatures. Check the overall heat balance.

Compare two modifications: doubling h, and replacing the cylinder by four cylinders of half the radius. Each smaller cylinder has the same length, conductivity, heat generation per unit volume and original heat transfer coefficient. The coolant temperature is unchanged.

See solution for Exercise 6

8 What you should now be able to calculate

Calculate heat loss and interface temperatures in plane and radial networks, and find a maximum temperature when heat is generated inside the solid.

Practice: Workshop 5 for resistance networks, heat generation and variable conductivity.

9 Report an error

To report an error, email Ranjeet Utikar. Include the title Heat transfer modes and steady conduction, the URL `/heat-transport/steady-conduction/`, and the equation or passage that needs correction.

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Solutions

Quiz solutions

Quiz 1. Heat transfer modes

Correct choice: 2.

The air is stationary, so there is no heat transport by bulk fluid motion. With radiation neglected, heat crosses the air layer by conduction. Heat transfer from the plate to the flowing water is convection. It combines conduction near the plate with heat transport by the moving water. Across a vacuum, heat transfers by radiation, which requires no material medium.

Back to Quiz 1

Quiz 2. Conductivity and the heat transfer coefficient

Correct choice: 3.

Turning up the fan changes the air flow over the plate. It does not change the metal or its temperature, so k stays the same. The coefficient h depends on the flow conditions and generally increases with air speed. More heat can therefore leave the plate for the same temperature difference between the plate and the air.

Back to Quiz 2

Quiz 3. Series and parallel networks

Correct choice: 4.

Heat passes through the outer lining, divides between the insulation and metal support, then passes through the inner lining. The heat rate through each lining is therefore the sum of the rates through the two middle paths. The insulation and support connect the same two uniform-temperature surfaces, so they have the same temperature drop. Their heat rates depend on their resistances and need not be equal.

Back to Quiz 3

Quiz 4. The largest temperature drop

Correct choice: 1.

The same heat rate passes through all five parts. Across each part,

$$|\Delta T_i| = |\dot{Q}| R_i$$

The insulation has the largest resistance, so it has the largest temperature drop. The steel has a very small resistance and therefore a small temperature drop.

Back to Quiz 4

Quiz 5. Radial heat rate and flux

Correct choice: 3.

At steady state, the heat entering the insulation must leave it. The same heat rate passes through every cylindrical surface:

$$\dot{Q}_r = 2\pi r L q_r = \text{constant}$$

Thus,

$$q_r = \frac{\dot{Q}_r}{2\pi r L}$$

As radius increases, the heat crosses a larger area. The heat flux therefore decreases even though the total heat rate stays the same.

Back to Quiz 5

Quiz 6. Centreline symmetry

Correct choice: 2.

There is no preferred radial direction at the axis. Symmetry requires

$$\left[\frac{dT}{dr}\right]_{r=0} = 0$$

Fourier's law then gives

$$q_r(0) = -k\left[\frac{dT}{dr}\right]_{r=0} = 0$$

The centre can still be hotter than the surface. Its temperature is a maximum, where the gradient is zero.

Back to Quiz 6

Quiz 7. Doubling the wire radius

Correct choice: 4.

At the axis, Eq. 11 gives

$$T_{\max} - T_0 = \frac{S_e R^2}{4k}$$

With heat generation and conductivity unchanged, doubling the radius makes this temperature difference four times as large:

$$(T_{\max} - T_0)_{\text{new}} = 2^2(12) = 48\ \text{K}$$

Back to Quiz 7

Quiz 8. Conductivity and diffusivity

Correct choice: 3.

Thermal diffusivity depends on conductivity, density and specific heat capacity:

$$\alpha = \frac{k}{\rho \hat{C}_p}$$

A higher conductivity increases this ratio, while a higher volumetric heat capacity reduces it. We need all three properties to compare the two alloys.

Back to Quiz 8

Quiz 9. Diffusion distance and time

Correct choice: 2.

The conduction distance is half the sheet thickness. It increases from 5 mm to 15 mm. For the same diffusivity,

$$\frac{t_{\text{new}}}{t_{\text{old}}} = \left(\frac{15}{5}\right)^2 = 9$$

Thus,

$$t_{\text{new}} = 9(50) = 450\ \text{s}$$

This is a diffusion time estimate. Finding the time to reach a particular temperature also requires the initial and surface conditions.

Back to Quiz 9

Quiz 10. Reducing the energy equation

Correct choice: 4.

Steady state removes accumulation. The wire is stationary, so there is no advection. Temperature is uniform around the circumference, so there is no angular conduction. Negligible end effects allow us to neglect axial conduction. Only radial conduction and internal heat generation remain, and they balance at steady state.

Back to Quiz 10

Exercise solutions

Exercise 1. Insulation around a spherical vessel

The given temperatures are at the insulation surfaces, so the required resistance is the insulation resistance alone. From Eq. 7,

$$R_{\text{sph}} = \frac{\frac{1}{0.50} - \frac{1}{0.55}}{4\pi(0.040)} \simeq 0.362\ \text{K W}^{-1}$$

The outward heat loss is

$$\dot{Q}_r = \frac{80 - 20}{R_{\text{sph}}} \simeq 166\ \text{W}$$

If the given temperatures were fluid temperatures, the fluid-film resistances would also be needed. For a vessel containing a cold fluid, the same resistance method gives the inward heat gain.

Back to Exercise 1

Exercise 2. Glass window: heat loss and inside surface temperature

Heat passes from the room air to the inner glass surface, through the glass, and then from the outer glass surface to the surroundings. These three steps form a series resistance network.

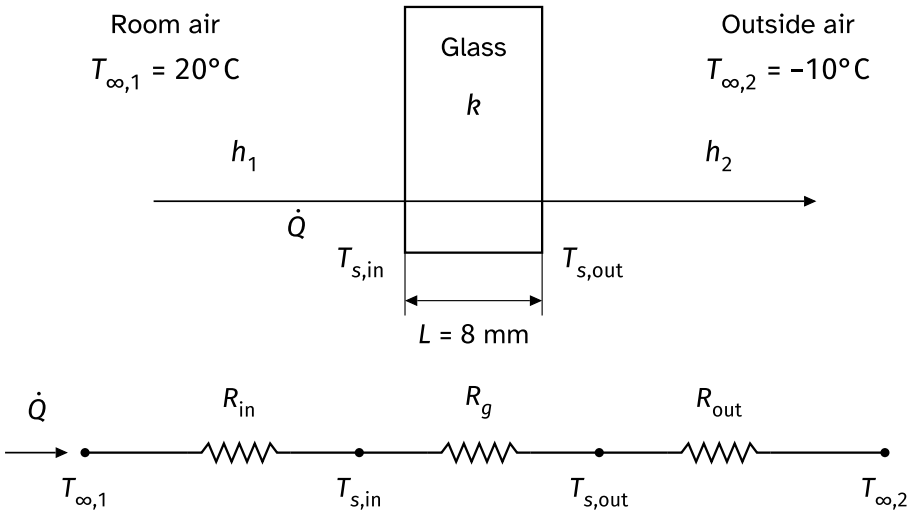


Figure 8: Heat flows from room air at 20 degrees Celsius through the inside film, glass and outside film to air at minus 10 degrees Celsius. The two glass surface temperatures are the unknown intermediate temperatures.

Data

Quantity	Symbol	Value
Window height	H	$0.8\ \mathrm{m}$
Window width	W	$1.5\ \mathrm{m}$
Glass thickness	L	$8\ \mathrm{mm} = 0.008\ \mathrm{m}$
Glass thermal conductivity	k	$0.78\ \mathrm{W\,m^{-1}\,K^{-1}}$
Room air temperature	$T_{\infty,1}$	$20\,^{\circ}\mathrm{C}$
Outside air temperature	$T_{\infty,2}$	$-10\,^{\circ}\mathrm{C}$
Inside heat transfer coefficient	h_1	$10\ \mathrm{W\,m^{-2}\,K^{-1}}$
Outside heat transfer coefficient, including radiation	h_2	$40\ \mathrm{W\,m^{-2}\,K^{-1}}$

A temperature difference of $1\,^{\circ}\mathrm{C}$ equals $1\ \mathrm{K}$. We can therefore express the supplied conductivity and heat transfer coefficients using kelvin-based units in the calculation.

We need the heat loss $\dot{Q}$ and the inside glass surface temperature $T_{s,\mathrm{in}}$.

Assumptions

- Steady state.
- Unidirectional heat flow, normal to the glass.
- Heat transfer through the edges and frame is neglected.
- No heat generation within the glass.
- Constant thermal conductivity and heat transfer coefficients.
- Uniform temperature on each glass surface.

The supplied outside heat transfer coefficient includes radiation, so no separate radiation resistance is added.

Calculations

First calculate the area normal to heat flow:

$$A = HW$$

Substituting the window dimensions gives

$$A = 0.8 \times 1.5 = 1.2\ \mathrm{m^2}$$

The inside film resistance is

$$R_{\mathrm{in}} = \frac{1}{h_1 A}$$

Using the inside coefficient and window area gives

$$R_{\text{in}} = \frac{1}{10 \times 1.2} = 0.083333\ \text{K W}^{-1}$$

For conduction through the glass,

$$R_g = \frac{L}{kA}$$

Substituting the thickness, conductivity and area gives

$$R_g = \frac{0.008}{0.78 \times 1.2} = 0.008547\ \text{K W}^{-1}$$

The outside film resistance is

$$R_{\text{out}} = \frac{1}{h_2 A}$$

Substituting the outside coefficient gives

$$R_{\text{out}} = \frac{1}{40 \times 1.2} = 0.020833\ \text{K W}^{-1}$$

The same heat rate passes through all three resistances, so they add in series:

$$R_{\text{total}} = R_{\text{in}} + R_g + R_{\text{out}}$$

Substituting the three resistances gives

$$R_{\text{total}} \simeq 0.112714\ \text{K W}^{-1}$$

The temperature difference between the air on either side is

$$\Delta T = T_{\infty,1} - T_{\infty,2}$$

Substituting the air temperatures gives

$$\Delta T = 20 - (-10) = 30\ \text{K}$$

The heat loss can now be calculated using Eq. 2. Writing that relation for this window gives

$$\dot{Q} = \frac{T_{\infty,1} - T_{\infty,2}}{R_{\text{total}}}$$

Substituting the temperature difference and total resistance gives

$$\dot{Q} = \frac{30}{0.112714} \simeq 266.16\ \text{W}$$

Heat flows from the warmer room to the colder outdoors. To find the inside glass temperature, consider the temperature drop across the inside film:

$$T_{\infty,1} - T_{s,\text{in}} = \dot{Q} R_{\text{in}}$$

The calculated heat rate gives an inside-film temperature drop of

$$\Delta T_{\text{in}} = 266.16 \times 0.083333 \simeq 22.180\,\text{K}$$

Rearranging the inside-film relation gives

$$T_{s,\text{in}} = T_{\infty,1} - \dot{Q} R_{\text{in}}$$

Subtracting the temperature drop from the room temperature gives

$$T_{s,\text{in}} = 20 - 22.180 = -2.180\,^{\circ}\text{C}$$

We can check this result by working from the outside air towards the room. The outer glass surface must be warmer than the outside air by $\dot{Q} R_{\text{out}}$:

$$T_{s,\text{out}} = T_{\infty,2} + \dot{Q} R_{\text{out}}$$

Substitution gives

$$T_{s,\text{out}} = -10 + 266.16 \times 0.020833 \simeq -4.455\,^{\circ}\text{C}$$

The temperature drop through the glass is therefore approximately

$$-2.180 - (-4.455) = 2.275\,\text{K}.$$

Dividing this drop by R_g recovers the heat rate of approximately 266 W. The surface temperatures also have the expected order: temperature falls from the room air to the inner glass surface, then to the outer surface and finally to the outside air.

Answer

The heat loss from the room through the window is

$$\dot{Q} \simeq 266\,\text{W}$$

The inside glass surface temperature is

$$T_{s,\text{in}} \simeq -2.18\,^{\circ}\text{C}$$

Most of the resistance is in the two fluid films, with the largest drop across the inside film. This explains why the glass surface is much colder than the room air. Adding a trapped air layer, as in double glazing, increases the resistance and reduces heat loss at the same air temperatures. The calculated inside surface is below freezing, so frost may form if sufficient moisture reaches it.

Back to Exercise 2

Exercise 3. Temperature and conservation in a heated wire

Calculation and answer

The electrical current produces heat at the volumetric rate $S_e = I^2/k_e = 10^6\,\text{W}\,\text{m}^{-3}$.

From Eq. 11, the centre-to-surface temperature difference is

$$\frac{10^6(0.002)^2}{4 \times 15} = 0.066667 \text{ K}.$$

The mean rise from Eq. 12 is half this value, 0.033333 K. Adding these rises to T_0 gives $T_{\max} = 25.0667\ ^\circ\text{C}$ and $\overline{T} = 25.0333\ ^\circ\text{C}$.

The surface flux is

$$q_r(R) = \frac{10^6 \times 0.002}{2} = 1000 \text{ W m}^{-2}.$$

Multiplying by the curved area

$$2\pi RL = 0.00628319 \text{ m}^2$$

gives $\dot{Q}_R = 6.28319$ W.

Independently, the total heat generation is $S_e \pi R^2 L = 6.28319$ W. This equals the surface heat loss, confirming the energy balance. The wire loses approximately 6.28 W at steady state.

Back to Exercise 3

Exercise 4. Temperature drops through a furnace wall

Heat passes through the hot-side film, firebrick, insulation, steel and cold-side film in series. We need the heat flux and the temperature at every solid interface.

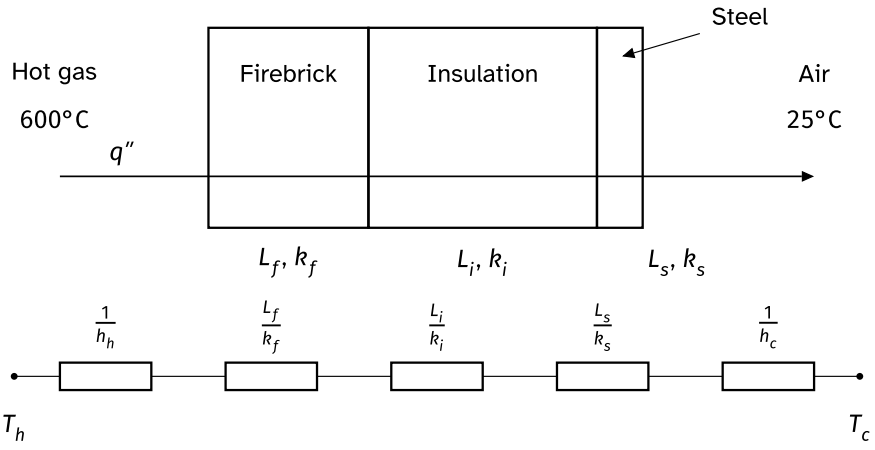


Figure 10: The furnace wall has three solid layers between hot gas and outside air. A common heat flux passes through all five resistance elements.

Data

Quantity	Symbol	Value
Hot-gas and cold-air temperatures	T_h, T_c	$600, 25\,^{\circ}\mathrm{C}$
Firebrick thickness and conductivity	L_f, k_f	$0.12\ \mathrm{m}, 1.2\ \mathrm{W\,m^{-1}\,K^{-1}}$
Insulation thickness and conductivity	L_i, k_i	$0.060\ \mathrm{m}, 0.050\ \mathrm{W\,m^{-1}\,K^{-1}}$
Steel thickness and conductivity	L_s, k_s	$0.0020\ \mathrm{m}, 16\ \mathrm{W\,m^{-1}\,K^{-1}}$
Hot-side and cold-side coefficients	h_h, h_c	$25, 10\ \mathrm{W\,m^{-2}\,K^{-1}}$

Assumptions

- Steady one-dimensional heat flow with constant properties.
- Perfect contact between the layers.
- No generation, edge loss or separate radiative transfer.

Calculations

We can calculate the heat flux directly by using $R'' = AR$, with units of $\mathrm{m^2\,K\,W^{-1}}$. The temperature drop is then $\Delta T = q'' R''$. For each solid layer, $R_i'' = L_i/k_i$; for each film, $R_h'' = 1/h$. In order from hot gas to cold air, these values are

Element	Resistance per unit area, $\mathrm{m^2\,K\,W^{-1}}$
Hot gas film, $1/h_h$	0.040
Firebrick, L_f/k_f	0.100
Insulation, L_i/k_i	1.200
Steel, L_s/k_s	0.000125
Cold air film, $1/h_c$	0.100

The total is $R''_{\mathrm{total}} = 1.440125\ \mathrm{m^2\,K\,W^{-1}}$. Dividing the gas-to-air temperature difference by this resistance gives

$$q'' = \frac{600 - 25}{1.440125} = 399.3\ \mathrm{W\,m^{-2}}$$

The temperature drop across each element is $q'' R_i''$. Subtracting these drops successively from the hot-gas temperature gives

- Hot gas: $600\,^{\circ}\mathrm{C}$.

- Hot firebrick surface: $584.0\ ^\circ\mathrm{C}$.
- Firebrick-insulation interface: $544.1\ ^\circ\mathrm{C}$.
- Insulation-steel interface: $64.98\ ^\circ\mathrm{C}$.
- Outer steel surface: $64.93\ ^\circ\mathrm{C}$.
- Cold air: $25\ ^\circ\mathrm{C}$.

The insulation has the largest temperature drop, $479.1\ \mathrm{K}$. The drops through the firebrick and steel are $39.93\ \mathrm{K}$ and $0.0499\ \mathrm{K}$, respectively. Since the heat flux is the same through all three layers, these differences are determined by L_i/k_i. The insulation has the largest resistance because of its combination of thickness and low conductivity.

Doubling the insulation thickness adds another $1.200\ \mathrm{m^2\,K\,W^{-1}}$ to the total resistance. The new heat flux is

$$q''_{new} = \frac{575}{1.440125 + 1.200} = 217.8\ \mathrm{W\,m^{-2}}$$

The heat loss decreases by 45.5%. It does not halve because the films, firebrick and steel still contribute the same resistances. For the original wall, adding all five temperature drops recovers the full $575\ \mathrm{K}$ difference between the gas and air.

Answer

The outward heat flux is $399\ \mathrm{W\,m^{-2}}$. The solid temperatures from the hot side are 584.0, 544.1, 64.98 and $64.93\ ^\circ\mathrm{C}$. Doubling the insulation thickness reduces the flux to $218\ \mathrm{W\,m^{-2}}$, a reduction of 45.5 percent.

The two cold-side interface temperatures retain guard digits to show the small contact-temperature drop.

Back to Exercise 4

Exercise 5. Insulation and boil-off from a spherical vessel

Warm air supplies heat through the outside film and spherical insulation. That heat evaporates nitrogen at its maintained boiling temperature.

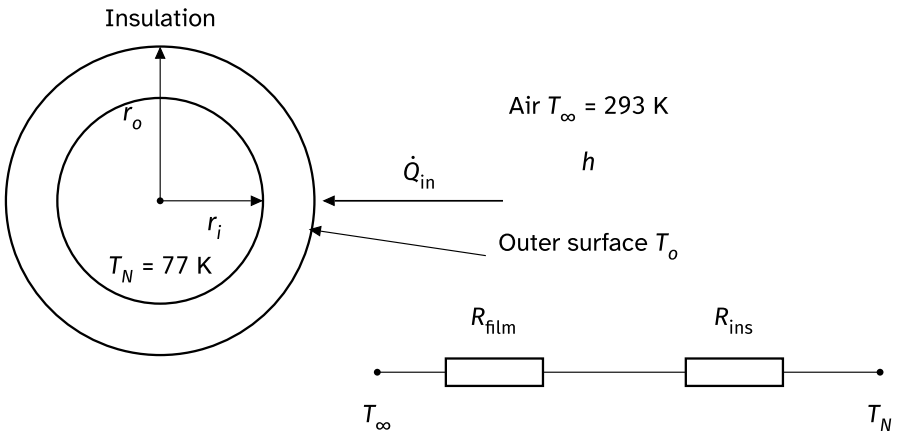


Figure 11: The cold spherical vessel has inner insulation radius ri and outer radius ro. Heat flows inward from ambient air through the outside film and insulation.

Data

Quantity	Symbol	Value
Vessel and outer insulation radii	r_i, r_o	$0.50, 0.54\ \mathrm{m}$
Liquid and air temperatures	T_N, T_∞	$77, 293\ \mathrm{K}$
Insulation conductivity	k	$0.040\ \mathrm{W\,m^{-1}\,K^{-1}}$
Outside coefficient	h	$10\ \mathrm{W\,m^{-2}\,K^{-1}}$
Nitrogen latent heat	h_{fg}	$198000\ \mathrm{J\,kg^{-1}}$

Assumptions

- Steady heat transfer with constant properties.
- Negligible shell resistance, support losses and radiation.
- The liquid remains at its boiling temperature.
- The same supplied outside coefficient is used for the insulated and bare comparisons.

Calculations

Heat flows from the warmer air through the outside film and then through the insulation to the nitrogen. The film and spherical conduction resistances are in series. Using Eq. 7 gives

$$R_{ins} = \frac{\frac{1}{0.50} - \frac{1}{0.54}}{4\pi(0.040)} = 0.2947\ \mathrm{K\,W^{-1}}$$

$$R_{film} = \frac{1}{10(4\pi)(0.54)^2} = 0.02729\ \mathrm{K\,W^{-1}}$$

Adding these resistances gives the heat leak:

$$\dot{Q}_{in} = \frac{293 - 77}{R_{ins} + R_{film}} = 670.8\ \mathrm{W}$$

The outside film uses the area at $r = 0.54\ \mathrm{m}$. The insulation resistance uses both radii to account for the changing area through the layer. A plane-wall expression with a single area would only approximate the spherical conduction resistance.

The heat entering the vessel evaporates nitrogen while its temperature remains constant. Dividing the heat rate by the latent heat gives

$$\dot{m}_{boil} = \frac{\dot{Q}_{in}}{h_{fg}} = \frac{670.8}{198000} = 0.003388\ \mathrm{kg\,s^{-1}} = 12.20\ \mathrm{kg\,h^{-1}}$$

The drop from the air to the outer insulation surface is $\dot{Q}_{in} R_{film}$. The surface temperature is therefore

$$T_o = 293 - \dot{Q}_{in} R_{film} = 274.7\ \mathrm{K},$$

between the air and liquid temperatures as expected for inward heat flow.

For the bare vessel, there is no insulation resistance. The outside film now acts over the spherical area at $r = 0.50\ \mathrm{m}$, giving

$$\dot{Q}_{bare} = 10(4\pi)(0.50)^2(293 - 77) = 6786\ \mathrm{W}$$

The resulting boil-off rate is

$$\dot{m}_{bare} = 123.4\ \mathrm{kg\,h^{-1}}$$

The insulation reduces both heat leak and boil-off by 90.1% for the specified conditions. This comparison uses the same h in both cases. If h were calculated from a convection correlation, changes in diameter and surface temperature could also affect its value.

Answer

With insulation, the heat leak is $671\ \mathrm{W}$, nitrogen boil-off is $12.2\ \mathrm{kg\,h^{-1}}$ and the outside insulation surface is at $274.7\ \mathrm{K}$. The bare vessel receives $6.79 \times 10^3\ \mathrm{W}$ and boils off $123\ \mathrm{kg\,h^{-1}}$.

Back to Exercise 5

Exercise 6. A heat-generating cylinder with an unknown surface temperature

The radial geometry is the same as the wire model in Figure 9. Here the surface temperature must be found from convection before the centre temperature can be evaluated.

Data

Quantity	Symbol	Value
Cylinder radius and length	R, L	$0.010, 1.0\ \mathrm{m}$
Thermal conductivity	k	$5.0\ \mathrm{W\,m^{-1}\,K^{-1}}$
Volumetric heat generation	S_e	$2.0 \times 10^6\ \mathrm{W\,m^{-3}}$
Coolant temperature	T_∞	$300\ \mathrm{K}$
Surface coefficient	h	$200\ \mathrm{W\,m^{-2}\,K^{-1}}$

Assumptions

- Steady radial conduction with uniform heat generation.
- Constant conductivity and convection coefficient.
- Negligible heat flow through the ends.
- Each comparison cylinder sees the same specified coolant temperature.

Calculations

The governing equation is the same as for the heated wire. At the axis, symmetry requires a zero radial temperature gradient:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{S_e}{k}=0$$

$$\left[\frac{dT}{dr}\right]_{r=0}=0$$

Integration with the symmetry condition gives

$$\frac{dT}{dr}=-\frac{S_e r}{2k}.$$

The surface temperature is now unknown, so we need a convection boundary condition at $r=R$. At steady state, the conductive flux reaching the surface equals the convective flux leaving it:

$$-k\left[\frac{dT}{dr}\right]_{r=R}=\frac{S_e R}{2}=h(T_s-T_\infty)$$

Rearranging gives

$$T_s-T_\infty=\frac{S_e R}{2h}.$$

Integrating the temperature gradient and using this surface temperature gives

$$T(r)=T_\infty+\frac{S_e R}{2h}+\frac{S_e}{4k}(R^2-r^2)$$

The two temperature rises come from different parts of the model. The surface rise follows from the overall generation-to-convection balance:

$$T_s-T_\infty=\frac{S_e R}{2h}=\frac{2.0\times10^6(0.010)}{2(200)}=50.0\ \mathrm{K}$$

The conduction profile then gives the additional centre rise:

$$T_{max}-T_s=\frac{S_e R^2}{4k}=\frac{2.0\times10^6(0.010)^2}{4(5.0)}=10.0\ \mathrm{K}$$

The surface-to-coolant temperature difference is $50.0\ \mathrm{K}$. The centre-to-surface difference is a further $10.0\ \mathrm{K}$. The temperatures are therefore $T_s=350.0\ \mathrm{K}$ and $T_{\max}=T(0)=360.0\ \mathrm{K}$.

The total heat generated is $S_e\pi R^2 L=628.3\ \mathrm{W}$. The surface heat loss is

$$h(2\pi RL)(T_s-T_\infty)=628.3\ \mathrm{W},$$

so the overall energy balance is satisfied. This check also confirms the surface boundary condition used to obtain the temperature profile.

Doubling h halves the surface-to-coolant difference to $25.0\ \mathrm{K}$. The centre-to-surface difference stays at $10.0\ \mathrm{K}$ because S_e, R and k are unchanged. The new centre temperature is

$$300 + 25 + 10 = 335.0\ \mathrm{K}.$$

Four cylinders of half the radius have the same total volume and heat generation as the original cylinder, but twice its total curved surface area. In each smaller cylinder, the surface-to-coolant difference is 25.0 K and the centre-to-surface difference is 2.50 K. The centre temperature is therefore

$$300 + 25 + 2.50 = 327.5\ \mathrm{K}.$$

The smaller radius reduces both temperature differences. At fixed S_e, h and k, the surface-to-coolant difference varies as R, whereas the centre-to-surface difference varies as R^2. These results assume that every smaller cylinder is exposed to the same coolant temperature and specified heat transfer coefficient.

Answer

The original cylinder has $T_s = 350.0$ K, $T_{max} = 360.0$ K and total heat loss 628 W. Doubling h lowers the centre to 335.0 K. Four cylinders of half the radius lower each centre to 327.5 K while preserving the original total heat generation.

Back to Exercise 6